PRINCIPLES OF ANALYSIS
TOPIC 3: COMPLETE ORDERED FIELDS

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Abstract. Ross declares fifteen axioms which are satisfied by the real numbers, and proceeds to prove the rest of the results in the book based on these axioms. We would like to know the extent to which the axioms characterize the real numbers. Specifically, are any to complete ordered fields "essentially the same", or is it possible there are two distinct fields, with different field structures, which could take the role of the real numbers?

Definition 1. A *field* is a set F together with binary operators

$$+ : F \times F \to F \quad \text{and} \quad \cdot : F \times F \to F$$

satisfying:

(A1) $a + (b + c) = (a + b) + c$ for all $a, b, c \in F$
(A2) $a + b = b + a$ for all $a, b \in F$
(A3) $\exists 0 \in F$ such that $a + 0 = a$ for all $a \in F$
(A4) $\forall a \in F \, \exists -a \in F$ such that $a + (-a) = 0$
(M1) $a(bc) = (ab)c$ for all $a, b, c \in F$
(M2) $ab = ba$ for all $a, b \in F$
(M3) $\exists 1 \in F$ such that $a \cdot 1 = a$ for all $a \in F$
(M4) $\forall a \in F \setminus \{0\} \, \exists a^{-1} \in F$ such that $aa^{-1} = 1$
(DL) $a(b + c) = ab + bc$ for all $a, b, c \in F$

Definition 2. Let E and F be fields. A *homomorphism* from E to F is a function $f : E \to F$ satisfying

(H1) $f(1_E) = 1_F$
(H2) $f(x + y) = f(x) + f(y)$
(H3) $f(xy) = f(x)f(y)$

Proposition 1. *Let $f : E \to F$ be a field homomorphism. Then f is injective.*

Definition 3. An *ordered field* is a field F together with a relation

$$\leq \subset F \times F$$

satisfying:

(O1) $a \leq b$ or $b \leq a$ for all $a, b \in F$
(O2) $a \leq b$ and $b \leq a$ implies $a = b$ for all $a, b \in F$
(O3) $a \leq b$ and $b \leq c$ implies $a \leq c$ for all $a, b, c \in F$
(O4) $a \leq b$ implies $a + b \leq b + c$ for all $a, b, c \in F$
(O5) $a \leq b$ and $0 \leq c$ implies $ac \leq bc$ for all $a, b, c \in F$

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Remark 1. Let F be an ordered field and let $x, y \in F$. Then

- $x < y$ means $x \leq y$ and $x \neq y$;
- $x \geq y$ means $y \leq x$;
- $x > y$ means $y < x$.

Proposition 2. *Let F be an ordered field.*
Then F contains a subfield isomorphic to $\mathbb{Q}$, which we identify with $\mathbb{Q}$.

Definition 4. Let F be an ordered field. Let $S \subset F$ and let $b \in F$.
We say that b is an *upper bound* for S if $s \leq b$ for every $s \in S$.
We say that b is a *lower bound* for S if $b \leq s$ for every $s \in S$.
We say that b is the *least upper bound (supremum)* of S, and write $b = \sup S$, if

(1) $s \leq b$ for every $s \in S$;
(2) if $s \leq c$ for every $s \in S$, then $b \leq c$.

We say that b is the *greatest lower bound (infimum)* of S, and write $b = \inf S$, if

(1) $b \leq s$ for every $s \in S$;
(2) if $c \leq s$ for every $s \in S$, then $c \leq b$.

Definition 5. Let F be an ordered field. We say that F is *complete* if

(CA) every subset of F which is bounded above has a least upper bound.

Proposition 3. *Let F be a complete ordered field. Then every subset of F which is bounded below has a greatest lower bound.*

Proposition 4 (Archimedean Property)**.** *Let F be a complete ordered field. Let $a, b \in F$ with $0 < a < b$. Then there exists $n \in \mathbb{N}$ such that $na < b$.*

We now given an example of an ordered field in which the Archimedean Property fails.

Recall from calculus that a *power series* around zero is a function of the form

$$\sum_{i=0}^{\infty} a_i x^i.$$

In complex analysis, one studies *Laurent series*, which are like power series which allow negative exponents. A general Laurent series is a function of the form

$$\sum_{i=n}^{\infty} a_i x^i,$$

where $n \in \mathbb{Z}$ is allowed to be a negative number. For example,

$$\frac{1}{x^2} + \frac{2}{x} + 3 + 4x + 5x^2 + 6x^3$$

is a Laurent series. A Laurent series is determined by the numbers a_i, which are the *coefficients*. We develop formal Laurent series by looking at sequences of coefficients.

Let F be an orderd field, and let

$$\mathcal{L}(F) = \{f : Z \to F \mid \text{ there exists } N \in \mathbb{Z} \text{ such that } f(n) = 0 \text{ for all } n < N\}.$$

Define addition on $\mathcal{L}(F)$ by

$$(f + g)(n) = f(n) + g(n).$$

Define multiplication on $\mathcal{L}(F)$ by

$$(fg)(n) = \sum_{i+j=n} f(i)g(j);$$

note that this is a finite sum.

Then $\mathcal{L}(F)$ together with these binary operations is a field. The additive identify is $f(n) = 0$, and the multiplicative identify is $f(n) = \delta_{n,0}$; here we use the *Kronecker delta*

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{otherwise.} \end{cases}$$

If $f = 0$, set $\operatorname{ord}(f) = 0$; otherwise, for $f \in \mathcal{L}(F)$, set $\operatorname{ord}(f) = \min\{f(n) \mid n \in \mathbb{N} \text{ and } f(n) \neq 0\}$; this exists for all nonzero f.

An element of $\mathcal{L}(f)$ is *positive* if $f(\operatorname{ord}(f)) > 0$. Define an order relation on $\mathcal{L}(F)$ by

$$f \leq g \quad \Leftrightarrow \quad f - g \text{ is negative} .$$

This endows $\mathcal{L}(F)$ the structure of an ordered field.

The real numbers embed in $\mathcal{L}(F)$ via the function $a \mapsto a\delta_{n,0}$, and we identify $\mathbb{R}$ with its image. Moreover, we identify a function with the corresponding Laurent series:

$$f \leftrightarrow \sum_{i=\operatorname{ord}(f)}^{\infty} f(i)x^i.$$

With this understanding, we see that $0 < x < a < \frac{1}{x}$ for every positive $a \in \mathbb{R}$. In particular, this field violates the Archimedean principle.

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